Reciprocal link for a three-component Camassa-Holm type equation

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Abstract

we discuss a reciprocal transformation for a three-component Camassa-Holm type equation and find that the transformed system is a reduction of the first negative flow for an extended MKdV hierarchy.

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1. Introduction

The Camassa-Holm (CH) equation

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1}$$

was derived by Camassa and Holm as a model for unidirectional dispersive shallow-water motion using an asymptotic expansion directly in the Hamiltonian for Euler's equations[1, 2]. It is completely integrable in the sense of Lax pair, bi-Hamiltonian structure, infinitely many conserved quantities etc. The equation can be solved by the inverse scattering transformation[3], and is found to admit multi-solitons solutions, algebro-geomrtric solutions and so on[4, 5, 6]. Especially, the discovery of peakons make the CH equation the subject of extensive research in recent years[1, 2]. The peakons is a weak solution in some Sobolov spaces and is interesting to research in water wave

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theory and in the view of general analysis for PDEs. Besides, the CH equation has a reciprocal link to the first negative flow of the KdV hierarchy[7], and the conservation laws of each may be connected[8]. Subsequently, many other CH type equations (the equations possess peakon solutions) are proposed and studied, such as the DP equation, the Novikov's equation and the Geng-xue equation[9, 10, 11, 12, 13, 14, 15]. Although many of them is integrable but it has some nonstandard features such as the weak Painlevé property, and is still much to understand via reciprocal transformation[16].

Recently, a three-component CH type system admitting the following 3×3 spectral problem

$$\psi_x = \begin{pmatrix} 0 & 1 & 0 \\ 1+\lambda v & 0 & u \\ \lambda w & 0 & 0 \end{pmatrix} \psi. \tag{2}$$

is proposed by Geng and Xue [17] with N-peakon solutions

$$\begin{aligned} u_t &= -vp_x + u_x q + \frac{3}{2}uq_x - \frac{3}{2}u(p_x r_x - pr), \\ v_t &= 2vq_x + v_x q, \\ w_t &= vr_x + w_x q + \frac{3}{2}wq_x + \frac{3}{2}w(p_x r_x - pr), \end{aligned} \tag{3}$$

where

$$\begin{aligned} u &= p - p_{xx}, \\ v &= \frac{1}{2}(q_{xx} - 4q + p_{xx}r_x - r_{xx}p_x + 3p_x r - 3pr_x), \\ w &= r_{xx} - r. \end{aligned}$$

The bi-Hamiltonian structure as well as infinite many conserved quantities of this three-component CH type equation and the dynamical system for the N-peakon solutions of it are also obtained[17, 18].

Very recently, another three-component CH type system [19] associated a 3×3 spectral problem

$$\phi_x = \begin{pmatrix} 0 & 0 & 1 \\ \lambda m_1 & 0 & \lambda m_3 \\ 1 & \lambda m_2 & 0 \end{pmatrix} \phi. \tag{4}$$

is constructed by us

$$
\begin{aligned}
&m_{1t}+u_2gm_{1x}-m_3(u_{2x}f-u_2g)-m_1(3u_2f-m_3u_2)=0,\\
&m_{2t}+u_2gm_{2x}+m_2(3u_{2x}g+m_3u_2)=0,\\
&m_{3t}+u_2gm_{3x}-m_3(2u_2f+u_{2x}g-m_3u_2)=0,
\end{aligned}
\tag{5}
$$

where

$$
m_i=u_i-u_{ixx},\ i=1,2,3,\quad f=u_3-u_{1x},\quad g=u_1-u_{3x}.
$$

However, a detail calculation shows the spectral (4) can be rewritten as (2), so above two systems can be contained in a same hierarchy. Bi-Hamiltonian structure and infinitely many conserved quantities for the CH type system (5) are worked out.

The propose of this paper is to construct a reciprocal transformation for the three-component CH type equation(5), and the relationship between the transformed system and an extended MKdV hierarchy is obtained. This will be done in Sec. 2 and Sec. 3.

2. A reciprocal transformation

In this section, we will construct a reciprocal transformation for the CH type system (5). As pointed out in Ref. [19], the three-component system (5) is the compatible condition of the spectral problem (4) and associated auxiliary problem

$$
\phi_t=\begin{pmatrix}
-u_2f & \lambda^{-1}u_2 & -u_2g\\
\lambda^{-1}f-\lambda m_1u_2g & u_2f+u_{2x}g-\lambda^{-2} & \lambda^{-1}g-\lambda m_3u_2g\\
-u_{2x}f & \lambda^{-1}u_{2x}-\lambda m_2u_2g & -u_{2x}g
\end{pmatrix}\phi. \tag{6}
$$

Based on above Lax pair, an infinite sequence of conservation laws may be constructed. Especially, one of them is

$$
((m_2m_3)^{\frac{1}{2}})_t=-((m_2m_3)^{\frac{1}{2}}u_2g)_x,
$$

which defines a reciprocal transformation

$$
dy=adx-au_2gdt,\quad d\tau=dt, \tag{7}
$$

where $a=(m_2m_3)^{\frac{1}{2}}$.

In the following, we will study the transformed system of (5). On the one hand, writing the column vector ϕ as $\phi = (\phi_1, \phi_2, \phi_3)^T$ and elimating ϕ_3 from the spectral problem (4), we get

$$\phi_{1xx} - \phi_1 = \lambda m_2 \phi_2, \quad \phi_{2x} = \lambda m_1 \phi_1 + \lambda m_3 \phi_{1x}.$$

which in the new variable can be written as

$$\phi_{1yy} + \frac{a_y}{a}\phi_{1y} - \frac{1}{a^2}\phi_1 = \lambda \frac{1}{m_3}\phi_2, \tag{8}$$

$$\phi_{2y} = \lambda \frac{m_1}{a}\phi_1 + \lambda m_3 \phi_{1y}. \tag{9}$$

Setting $b = e^{-\partial_y^{-1} \frac{m_1}{a m_3}}$, by making a gauge transformation

$$\phi_1 = b\varphi_1, \quad \phi_2 = m_3 b \varphi_2,$$

the spectral problem (8)-(9) is transformed to

$$\varphi_{1yy} - Q_2 \varphi_{1y} - Q_1 \varphi_1 = \lambda \varphi_2, \tag{10}$$

$$\varphi_{2y} = \lambda \varphi_{1y} + Q_3 \varphi_2, \tag{11}$$

where

$$Q_1 = -\frac{b_{yy}}{b} - \frac{a_y b_y}{ab} + \frac{1}{a^2}, \quad Q_2 = -2\frac{b_y}{b} - \frac{a_y}{a}, \quad Q_3 = \frac{m_1}{a m_3} - \frac{m_{3y}}{m_3}.$$

It is easy to find that the scalar form of the spectral problem (10)-(11) is

$$\partial_y^2 + u\partial_y + v + \partial_y^{-1} w, \tag{12}$$

which is the spectral problem of an extended modified KdV hierarchy[20], and this extended MKdV hierarchy is related to the extended KdV hierarchy(the Yajima-Oikawa hierarchy[21, 22]) via a Miura transformation[20]. Furthermore, as we already mentioned[13], the transformation between (Q_1, Q_2, Q_3) and (u, v, w) may be also connected via a Miura transformation. Indeed, the Miura transformation

$$u = -(Q_2 + Q_3), \quad v = Q_2 Q_3 - Q_1 + Q_{3y}, \quad w = Q_1 Q_3 - (Q_2 Q_3)_y - Q_{3yy} \tag{13}$$

is obtained from factorizing the Lax operator (12), that is

$$\partial_y^2 + u\partial_y + v + \partial_y^{-1} w = \partial_y^{-1}(\partial_y - Q_3)(\partial_y^2 - Q_2\partial_y - Q_1).$$

Moreover, the spectral problem (10)-(11) is reduced to the spectral problem of the MKdV hierarchy as $Q_1 = Q_2 = 0$, and we called the hierarchy associated with this spectral problem an extend MKdV hierarchy too.

Similarly, the auxiliary problem (6) may be changed to the following form

$$\varphi_{1\tau} = \lambda^{-1} q_3 \varphi_2,$$
$$\varphi_{2\tau} = (-\lambda^{-2} + q_3)\varphi_2 + \lambda^{-1} q_2 \varphi_{1y} + \lambda^{-1} q_1 \varphi_1,$$

where

$$q_1 = \frac{u_3 - u_{1y}a}{m_3} - \frac{m_1}{m_3^2}(u_1 - u_{3y}a), \quad q_2 = \frac{a}{m_3}(u_1 - u_{3y}a), \quad q_3 = m_3 u_2.$$

Then the Lax pair for the three-component CH type system (5) in the new variable may be converted into

$$\varphi_y = \begin{pmatrix} 0 & 1 & 0 \\ Q_1 & Q_2 & \lambda \\ 0 & \lambda & Q_3 \end{pmatrix} \varphi, \tag{14}$$

and

$$\varphi_\tau = \begin{pmatrix} 0 & 0 & \lambda^{-1} q_3 \\ 0 & q_3 & \lambda^{-1}(q_{3y} + Q_3 q_3) \\ \lambda^{-1} q_1 & \lambda^{-1} q_2 & -\lambda^{-2} + q_3 \end{pmatrix} \varphi. \tag{15}$$

On the other hand, we can now to state the transformed system for (5). Under the transformation (7), the CH type system (5) is changed to

$$\begin{aligned} m_{1\tau} &= m_1(3u_2 f - m_3 u_2) + m_3(u_{2y} a f - u_2 g), & m_1 &= u_1 - a(u_{1y}a)_y, \\ m_{2\tau} &= -m_2(3u_{2y} a g + m_3 u_2), & m_2 &= u_2 - a(u_{2y}a)_y, \\ m_{3\tau} &= m_3(2u_2 f + u_{2y} a g - m_3 u_2), & m_3 &= u_3 - a(u_{3y}a)_y. \end{aligned}$$

Through tedious but direct calculations, we get the system for Q_1, Q_2, Q_3 under the Liouville transformation

$$\begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} = \begin{pmatrix} (m_2 m_3)^{-1}((\frac{m_1}{m_3})_x + 1 - (\frac{m_1}{m_3})^2) \\ (m_2 m_3)^{-\frac{3}{2}}(2m_1 m_2 - \frac{1}{2}(m_2 m_3)_x) \\ m_2^{-\frac{1}{2}} m_3^{-\frac{3}{2}}(m_1 - m_{3x}) \end{pmatrix}, \quad y = \int_{-\infty}^{x} (m_2 m_3)^{\frac{1}{2}} dx,$$

that is

$$\begin{aligned} Q_{1\tau} &= Q_1 q_3 - q_1, & S_1 &= ((\partial + Q_3 - Q_2)(\partial + Q_3) - Q_1) q_3 = -1, \\ Q_{2\tau} &= 2q_{3y} + Q_3 q_3 - q_2, & S_2 &= (-\partial + Q_3) q_1 - Q_1 q_2 = 0, \\ Q_{3\tau} &= -Q_3 q_3 + q_2, & S_3 &= (-\partial + Q_3 - Q_2) q_2 - q_1 = -1. \end{aligned} \tag{16}$$

Direct calculation shows that the compatible condition of the Lax pair (15) is just the transformed system (16), therefore under the reciprocal transformation (7), the three-component CH type system(5) and its Lax pair are changed to which of (16) accordingly.

3. The relation between the transformed CH type system and an extended MKdV hierarchy

According to Ref. [20], the extended MKdV hierarchy associated with the spectral problem (12) may be formulated as a bi-Hamiltonian system admitting the following Hamiltonian pair

$$
\begin{aligned}
\mathcal{J}_1 &= \begin{pmatrix} 0 & 0 & 2\partial_y \\ 0 & 2\partial_y & \partial_y^2 + u\partial_y \\ 2\partial_y & -\partial_y^2 + \partial_y u & 0 \end{pmatrix}, \\
\mathcal{J}_2 &= \begin{pmatrix} 6\partial_y & * & * \\ 4u\partial_y & 2\partial_y^3 + 2u\partial_y u + \partial_y v + v\partial_y & * \\ 2\partial_y^3 - 2\partial_y u\partial_y + 2v\partial_y & \theta_1 & \theta_2 \end{pmatrix},
\end{aligned}
$$

where

$$
\begin{aligned}
&\theta_1 = -\partial_y^4 + \partial_y^3 u + \partial_y u\partial_y^2 - \partial_y u\partial_y u - v\partial_y^2 + v\partial_y u + 2w\partial_y + \partial_y w, \\
&\theta_2 = \partial_y uw + uw\partial_y + w\partial_y^2 - \partial_y^2 w.
\end{aligned}
$$

Based on the Miura link (13), the compatible Hamiltonian operators for another extend MKdV hierarchy of $Q = (Q_1, Q_2, Q_3)^T$ may be obtained, which are

$$
\tilde{\mathcal{J}}_i = \mathcal{F}^{-1}\mathcal{J}_i\mathcal{F}^{-1*},\ i = 1, 2, \qquad \mathcal{F} = \begin{pmatrix} 0 & -1 & -1 \\ -1 & Q_3 & \partial_y + Q_2 \\ Q_3 & -\partial_y Q_3 & Q_1 - \partial_y Q_2 - \partial_y^2 \end{pmatrix}.
$$

Then the first negative flow for this extend MKdV hierarchy is obtained, that is

$$
\begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}_\tau = \mathcal{F}^{-1}\mathcal{J}_1 \begin{pmatrix} A \\ B \\ C \end{pmatrix}, \qquad \mathcal{F}^{-1}\mathcal{J}_2 \begin{pmatrix} A \\ B \\ C \end{pmatrix} = 0. \tag{17}
$$

In ordering to get the relationship between the transformed system (16) and the first negative flow of the extend MKdV hierarchy for Q, we introduce

$$\begin{aligned} A &= \frac{1}{4}(Q_2+Q_3)\partial_y^{-1}(S_1-S_3) - \frac{1}{2}\partial_y^{-1}S_2 + \frac{1}{4}(S_1+S_3) - Q_3q_{3y} - Q_3^2q_3, \\ B &= \frac{1}{2}\partial_y^{-1}(S_1-S_3) - Q_3q_3, \\ C &= -q_3, \end{aligned} \tag{18}$$

(here all integration constants are assumed to be zero). Substituting (18) into (17), we arrive at the following system for the first negative flow

$$\begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}_\tau = \begin{pmatrix} Q_1q_3 - q_1 \\ 2q_{3y} + Q_3q_3 - q_2 \\ -Q_3q_3 + q_2 \end{pmatrix}, \quad \mathcal{F}^{-1}\mathcal{K}\begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix} = 0, \tag{19}$$

where

$$\mathcal{K} = \begin{pmatrix} \frac{1}{2}\partial_y u\partial_y^{-1} - \frac{1}{2}\partial_y & -3 & \frac{3}{2}\partial_y - \frac{1}{2}\partial_y u\partial_y^{-1} \\ Q_3\partial_y + v + \frac{1}{2}v_y\partial_y^{-1} & -2u & u\partial_y - \partial_y^2 - v - \frac{1}{2}v_y\partial_y^{-1} \\ \chi_1 & \partial_y u - \partial_y^2 - v & \chi_2 \end{pmatrix},$$

herein

$$\chi_1 = -Q_3\partial_y^2 - (2Q_{3y} + Q_2Q_3)\partial_y + \frac{1}{2}(3w + w_y\partial_y^{-1}),$$

$$\chi_2 = \partial_y^3 + v\partial_y - \partial_y u\partial_y - \frac{1}{2}(3w + w_y\partial_y^{-1}).$$

and (u, v, w) are given by (13).

Noticing that the operator ∂_y^{-1} comes from the definition of A, B, C, then if $S_1 = S_3 = -1$ and $S_2 = 0$, we have $\mathcal{K}(S_1, S_2, S_3)^T = 0$, so the transformed three-component CH type system (16) is a reduction of the first negative flow for the extended MkdV hierarchy possessing the spectral problem (14).

Remark. The relationship of Hamiltonian structures between the three-component CH type system (5) and the transformed system (16) may be obtained following the step in Ref. [23] with the help of the Liouville transformation.

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